RIBS-PH-4/97
q-alg/9703032

Green Function on the q-Symmetric Space $SU_q(2)/U(1)$

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Abstract

Following the introduction of the invariant distance on the non-commutative C-algebra of the quantum group $SU_q(2)$, the Green function and the Kernel on the q-homogeneous space $M = SU(2)_q/U(1)$ are derived. A path integration is formulated. Green function for the free massive scalar field on the non-commutative Einstein space $R^1 \times M$ is presented.

March 1997

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1 Introduction

How the quantum mechanical effects should be altered if we replace the space-time continuum with a non-commutative geometry is an exciting question. To answer this question we have to formulate the known quantum mechanical problems over the non-commutative spaces. Since we lack satisfactory mathematical tools, construction of the Schrödinger equations over the non-commutative spaces is difficult and sometimes arbitrary [1]. First of all when we do not have a differentiable manifold it is problematic to find the correct operators replacing the derivatives. If however the non-commutative geometry is given as a quantum group space this problem may be solved in a natural way for having the action of the q-algebra generators one is not required to deal with q-differential calculus. If this is not the case, since it is always possible to build up an integration theory on a given set, the path integrals may in principle be the suitable method of quantization for the non-commutative geometries in general. Therefore the derivation of the Green functions over the non-commutative spaces is of interest. We should also remember that in the usual quantum physics defined over the commutative spaces the Green functions which are the vacuum (or temperature) expectation values of two point field operators play important role [2]. Even for the q-group spaces the construction of the Green functions seems to be important step in the formulation of many q-deformed quantum mechanical problems.

The experience we have in the derivation of Green functions over the (undeformed) group manifolds is quite rich; and it is also well known that many non-relativistic quantum mechanical problems are related to the particle motion over these manifolds [3]. Therefore we hope that constructing the Green functions on the q-group spaces may lead meaningful definitions of these problems over the non-commutative geometries.It is important to stress that if we know the formulations of the non-relativistic potential problems we also gain insight in some field theoretical effects. In fact, the calculations of many field theoretical problems like the pair creations in the given cosmologies or in the external electromagnetic fields, and the Casimir interactions may formally become equivalent to some non-relativistic potential problems. For example to investigate the pair production in the Robertson-Walker space-time expanding with the factor $a(t)$ one has to calculate the Green function of a particle moving in the one-dimensional potential $V(t) = a^{-2}(t)$ with the

time t playing the role of coordinate [4].

Motivated by the considerations summarized above the construction of the Green functions over the quantum group spaces is the subject of this work. The specific example we study is the quantum symmetric space $M = SU_q(2)/U(1)$.Quantum symmetric spaces have already been subject of some interesting investigations. For example the well known relations between the special functions and the classical groups have been generalized to the quantum groups [5]; and quantum spheres are studied [6]. Recently the homogeneous space of $E_q(2)$ is considered and q-Schrödinger equation on it is constructed [7].

In section II after a brief review of the invariant distance concept, we outline a method for constructing the Green functions by two classical group examples.In this method, which is applicable to the quantum groups, one first constructs the one-point "Green function", then obtains the Green function depending on two points by the group action.

In Section III the invariant distance for the quantum group $A = Pol(SU_q(2))$ and the q-symmetric space $M = SU_q(2)/U(1)$ is constructed and its properties are demonstrated.

In Section IV the one-point "Green function" is derived on the space M, from which we obtain the Green function in section V.

In Section VI we introduce the time development Kernel on the space M. Having this Kernel in hand the non-commutative path integration is also formulated.

Finally in Section VII the Green function for the massive scalar field on $R^1 \times M$ which is the non-commutative version of the Einstein space is constructed.

The basic definitions and the established results about the Hopf algebra A which we use in our work are given in the Appendix.

2 Method for Constructing Green Functions. Examples from the classical Lie groups

The Green function of the free particle motion over a Lie group manifold and its homogeneous spaces depend on the invariant distance between two points. When we attempt to construct Green functions over the quantum group spaces the first problem we have to face is the introduction of the invariant distance. To overcome this problem it is instructive to review the case of the classical Lie groups for the purpose of developing a method for constructing the Green functions which can also be employed for the quantum groups.We briefly study two examples.

We first consider the real line $\mathcal{R}$ which is the homogeneous space with respect to the translation group $T(x)$. The Green function of the free particle motion over $\mathcal{R}$ depends on the invariant distance $\mid x - x' \mid$ and on the momentum $p \in (-\infty, \infty)$ which is the weight of the unitary irreducible representation. The homogeneity of $\mathcal{R}$ under the action of $T(x)$ implies

$$\mathcal{G}^p(x, x') = T^{-1}(x')\mathcal{G}^p(x, 0). \tag{II.1}$$

The above equation suggests that the invariant Green function on the homogeneous space can be obtained by the group action if the one-point "Green function" $\mathcal{G}^p(x, 0)$ is known.

As the second example we consider the classical symmetric space $SU(2)/U(1)$. If an element $g \in SU(2)$ is parametrized as $g = kak'$ with

$$k = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}, a = \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{II.2}$$

the symmetric space M which is topologically equivalent to S^2 is represented as

$$M = g\sigma(g^{-1}). \tag{II.3}$$

Here σ is the involutive automorphism having the property

$$\sigma(k) = k, \qquad \sigma(a) = a^{-1}. \tag{II.4}$$

The Green function $\mathcal{G}$ on M depends on the group invariants which are the weight of the irreducible representation l=0,1,2,... and on the invariant distance given by

$$\rho(x_1, x_2) = 1 - \frac{1}{2}Tr(x_1 x_2^{-1}) \tag{II.5}$$

with $x_1, x_2 \in M$. Using the group property $g_2^{-1} g_1 = g_{12}$ we can write

$$Tr(x_1 x_2^{-1}) = Tr(g_1 \sigma(g_1^{-1})(g_2 \sigma(g_2^{-1}))^{-1}) = Tr(g_2^{-1} g_1 \sigma((g_2^{-1} g_1)^{-1})) = \\ = Tr(g_{12} \sigma(g_{12}^{-1})) = Tr(x_{12}). \quad \text{(II.6)}$$

If we fix one of the points as $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain a formula depending on only one point. Taking the advantage of this formula we can first construct a one-point "Green function", then by the action of the group element we arrive at the Green function which is dependent on two points. The equation satisfied by the one-point "Green function" is

$$(\mathcal{C} - (l + 1/2)^2)\mathcal{G}^l(x) = \delta(x) \quad \text{(II.7)}$$

where $\mathcal{C}$ is the center of the enveloping algebra $U(su(2))$. Once we obtain the solution of the above equation we can derive the Green function simply by the group action as

$$\mathcal{G}^l(x_1, x_2) = T(g_2^{-1})\mathcal{G}^l(x_1) = \mathcal{G}^l(g_2 x_1 \sigma(g_2^{-1})) \quad \text{(II.8)}$$

which satisfies

$$(\mathcal{C} - (l + 1/2)^2)\mathcal{G}^l(x_1, x_2) = \delta(x_1 - x_2). \quad \text{(II.9)}$$

3 An Invariant Distance on the Quantum Group $SU_q(2)$

The coordinate functions a, a^*, b, b^* of the Hopf algebra A (see App.A) satisfy the commutation relations [6]:

$$b^* b = b b^*, \quad ba = qab, \quad b^* a = qab^*, \\ aa^* + b^* b = 1, \quad a^* a + q^2 b b^* = 1 \quad \text{(III.1)}$$

The $*$-representation of the quantum group A (the representation of the C-algebra) in the Hilbert space $\mathcal{H}$ with the orthonormal basis $\{\mid n\rangle\}$, n=0,1,2,... with $q < 1$ is given by [6]

$$a \mid n\rangle = (1 - q^{2n})^{1/2} \mid n-1\rangle, \qquad b \mid n\rangle = e^{i\phi} q^n \mid n\rangle, \\ b^* \mid n\rangle = e^{-i\phi} q^{n+1} \mid n\rangle, \qquad a^* \mid n\rangle = (1 - q^{2n+2})^{1/2} \mid n+1\rangle \quad \text{(III.2)}$$

is irreducible for fixed $\phi \in (0, 2\pi)$.

In $q \to 1$ limit the relations (III.2) define the three dimensional space which is the manifold of $SU(2)$.To understand the situation in the deformed case we recall the usual quantum mechanics in which the physical systems are defined by the vectors in the Hilbert space and the self adjoint operators correspond to the observables including coordinates. In the same manner for quantum groups we construct self-adjoint operators X from the linear combinations of the coordinate functions given in (III.2) and define the expectation values of these operators as the points of the quantum group space as

$$X_\psi = \langle \psi \mid X \mid \psi \rangle, \quad X \in A, \psi \in \mathcal{H}. \tag{III.3}$$

The above definition can be carried to co-product space $X \otimes Y$ with $X, Y \in A$ to define the invariant distance on $SU_q(2)$ which should go to the corresponding classical limit as $q \to 1$ and should have the following properties:

$$\rho(g_1, g_2) = \rho(g_2, g_1), \tag{III.4}$$

$$\rho(g_1 g, g_2 g) = \rho(g_2, g_1), \tag{III.5}$$

$$\rho(g_1, g_2) > 0, \tag{III.6}$$

$$\rho(g, g) = 0, \tag{III.7}$$

$$\rho(g_1, g_2) < \rho(g_1, g_3) + \rho(g_3, g_2). \tag{III.8}$$

Motivated by formula (II.5) of the previous section we suggest the following hermitian operator in $A \otimes A$:

$$R = (\tau \otimes S)\Delta(1 - \frac{1}{[2]_q} Tr_q(d^{1/2})) \tag{III.9}$$

Here $d^{1/2}$ is the matrix of the unitary irreducible corepresentation of the Hopf algebra A with weight 1/2 (see App.A). $[\cdot]_q$ is define as $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$ and the q-trace is given by

$$Tr_q(d^{1/2}) = \sum_{j=-1/2}^{1/2} q^{-2j} (d^{1/2}_{jj}). \tag{III.10}$$

τ is the automorphism of A defined as

$$\tau(d^{1/2}) = \begin{pmatrix} q^{-1} a & b \\ -q b^* & q a^* \end{pmatrix}. \tag{III.11}$$

Expectation value of the operator R of (III.9) in the Hilbert space $H \otimes H$ defines the correct invariant distance for A. In fact this operator possesses all the properties of (III.4–III.8).

(i) Symmetry condition of (III.4) is fulfilled for

$$\sigma R = R. \tag{III.12}$$

Here σ is the flip homomorphism $\sigma(x \otimes y) = y \otimes x$; $x, y \in A$.

(ii) Invariance condition of (III.5) takes the form of

$$\langle \Delta \otimes \Delta R \rangle_4 = R \tag{III.13}$$

where $\langle \cdot \rangle_4$ is the map from $A \otimes A \otimes A \otimes A$ into $A \otimes A$ is given by

$$\langle a \otimes b \otimes c \otimes d \rangle_4 = (a \otimes c)\phi(b\tau^{-1}(d)) \tag{III.14}$$

with $\phi(x)$ being the invariant integral on the quantum group A (see app.B).

(iii) The operator R is positive. To show this we have to construct the basis in the Hilbert space $H \otimes H$ in which it is diagonal. We choose the eigenfunctions as

$$\psi_l = \sum_{j=-l}^{l} v_j^l \mid l+j\rangle \otimes \mid l-j\rangle. \tag{III.15}$$

We then sandwich the operator R of (III.9)between the above states to get the recurrence relations for the unknown coefficients v_j^l

$$[2]_q v_j^l (1 - \frac{2q^{2l+1}}{[2_q]} - E_l) = v_{j-1}^l [l+j][l-j+1] + v_{j+1}^l [l-j][l+j+1]. \tag{III.16}$$

These coefficients are normalized as

$$\sum_{j=-l}^{l} \overline{v_j^l} v_j^l = 1. \tag{III.17}$$

E_l in (III.16) is the spectrum of the self-adjoint operator R and $[x]$ is defined as $[x] = \sqrt{1 - q^{2x}}$. As an example consider the eigenvalue and eigenfunction for $l = 0$ state

$$\psi_0 = \mid 0\rangle \otimes \mid 0\rangle. \tag{III.18}$$

The corresponding eigenvalue

$$E_0 = \frac{q^{-1} - q}{q^{-1} + q} \tag{III.19}$$

is positive for $q < 1$. Similar demonstrations can be done for all other values of l to prove that E_l is positive.

(iv) condition of (III.7) is also satisfied for

$$m(\tau^{-1} \otimes id)R = 0 \tag{III.20}$$

where m is the operation of multiplication in the C-algebra and τ^{-1} is the inverse of the involution (III.11).

(v) Finally the triangular inequality of (III.8) reads

$$\langle \psi \mid (id \otimes \sigma)(R \otimes 1) \mid \psi \rangle < \langle \psi \mid (R \otimes 1 + 1 \otimes R) \mid \psi \rangle, \quad \mid \psi \rangle \in H \otimes H \otimes H. \tag{III.21}$$

As in the case (iii) we can show that the self-adjoint operator Π which is defined as

$$\Pi = R \otimes 1 + 1 \otimes R - (id \otimes \sigma)(R \otimes 1) \tag{III.22}$$

is positive. For example for the eigenfunction

$$\psi_0 = \mid 0 \rangle \otimes \mid 0 \rangle \otimes \mid 0 \rangle, \quad \langle \psi_0 \mid \psi_0 \rangle = 1 \tag{III.23}$$

the corresponding eigenvalue which is given by

$$E_0 = \frac{q^{-1} - q}{q^{-1} + q} \tag{III.24}$$

is positive for $q < 1$.

Before closing this section we give the invariant distance on the coset space $M = A/K$ where $K = Pol(U(1))$(see App. B). We first introduce the involutive automorphism

$$\beta(d^{1/2}) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \tag{III.25}$$

It is clear that it does not change the quantum subgroup K (see App. A). By virtue of this automorphism we introduce the coordinate functions on the q-symmetric space $M = A/K$ as

$$t_{ij} = (d^{1/2} \beta(S(d^{1/2})))_{ij}. \tag{III.26}$$

The operator of the invariant distance on M is then given by

$$R_M = (\tau \otimes S)\Delta(1 - \frac{1}{[2]_q} Tr_q(t)). \tag{III.27}$$

Introducing (III.26) into the above formula we obtain

$$R_M = (\tau \otimes S)\Delta\xi \tag{III.28}$$

where $\xi = -q^{-1}bc$ is the element of the two-sided coset space $H = K/A/K$ (see App.A).

4 One-Point "Green Function"

We will follow the method we introduced in Sec.II in the construction of the Green function over M. We first have to obtain the one-point "Green function" $\mathcal{G}_q^l(\xi)$ .It is defined by the deformation of (II.7) as

$$(\mathcal{C} - [l + \frac{1}{2}]^2)\mathcal{G}_q^l(\xi) = \delta_q(\xi) \tag{IV.1}$$

where $\mathcal{C}$ is the center of the Hopf algebra $U(su_q(2))$ (see App.C). The invariant q-delta function $\delta_q(\xi)$ is a linear functional over the two-sided coset space H which for any function $f \in A[0,0]$ satisfies

$$\langle \delta_q(\xi) \mid f(\xi) \rangle = f(0), \quad f \in A[0,0] \tag{IV.2}$$

where scalar product is the one given in App.B. It is easy to verify that the q-delta function can be represented as

$$\delta_q(\xi) = \sum_{l=0}^{\infty} [l + \frac{1}{2}] d_{0,0}^l(\xi) \tag{IV.3}$$

where $d_{0,0}^l(\xi)$ is the q-zonal function [6]

$$d_{0,0}^l(\xi) = \phi_{2,1}(q^{-2l}, q^{2(l+1)}, q^{2l} \mid q^2, q\xi). \tag{IV.4}$$

We can also verify by direct substitution that the one-point "Green function" of (IV.1) can be represented as

$$\mathcal{G}_q^l(\xi) = \sum_{n=0}^{\infty} [n + \frac{1}{2}] \frac{D_{0,0}^l(\xi)}{[n + \frac{1}{2}]^2 - [l + \frac{1}{2}]^2}. \tag{IV.5}$$

The above summation can be executed to give an expression in terms of the q-hypergeometric function:

$$\mathcal{G}_q^l(\xi) = \gamma^l \xi^{-l-1} \phi_{2,1}(q^{-2(l+2)}, q^{-2(l+2)}, q^{-4(l+2)} \mid q^{-2}, q^{-2}\xi^{-1}) \tag{IV.6}$$

where γ^l is the normalization constant

$$\gamma^l = \frac{1}{[l][l+1]}(\int_0^1 d_q\xi \mathcal{G}_q^l(\xi))^{-1} = \frac{\sqrt{q}}{(-1-l)_{-q}} (\phi_{2,1}(q^{-2(l+2)}, q^{-2l)}, q^{-4(l+2)} \mid q^{-2}, q^{-2}))^{-1} \tag{IV.7}$$

with $(x)_q = \frac{1-q^x}{1-q}$.

5 Green Function over $M = A/K$

Having the one-point Green function in hand we can now introduce the Green function $\mathcal{G}_q^l(M \otimes M)$ on the q-symmetric space $M = A/K$ as

$$\mathcal{G}_q^l(M \otimes M) = (\tau \otimes S)\Delta \mathcal{G}_q^l(\xi). \tag{V.1}$$

The equation satisfied by the above Green function is

$$\left(id \otimes \{\mathcal{C} - [l + \frac{1}{2}]^2\}\right) \circ \mathcal{G}_q^l(M \otimes M) = \delta_q(M \otimes M) \tag{V.2}$$

where the invariant q-delta function is given by

$$\delta_q(M \otimes M) = (\tau \otimes S)\Delta \delta_q(\xi). \tag{V.3}$$

Substituting (IV.3) into the above equation and (IV.5) into (V.2) we obtain the following representations for the invariant q-delta function and the Green function:

$$\delta_q(M \otimes M) = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} [l + \frac{1}{2}] \tau(d_{0,j}^l(M)) \otimes \overline{d_{0,j}^l}(G). \tag{V.4}$$

$$\mathcal{G}_q^l(M \otimes M) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} [n + \frac{1}{2}] \frac{\tau(d_{j,0}^n(M)) \otimes \overline{d_{0,j}^n}(M)}{[l + \frac{1}{2}]^2 - [n + \frac{1}{2}]^2} \tag{V.5}$$

Using the representation (IV.6) of the one-point "Green function" we have another expression for $\mathcal{G}_q^l(M \otimes M)$ in terms of the q-hypergeometric function as

$$\mathcal{G}_q^l(M \otimes M) = \gamma^l(\tau \otimes S)\Delta(\xi^{-l-1}\phi_{2,1}(q^{-2(l+2)}, q^{-2(l+2)}, q^{-4(l+2)} \mid q^{-2}, q^{-2}\xi^{-1})) \tag{V.6}$$

For any operator function $f(M) \in M$ and the linear operator P of the dual Hopf algebra $U(su_q(2))$ (see App.C) the q-delta function of (V.4) satisfies

$$\langle \delta_q(M \otimes M) \mid id \otimes f \rangle_2 = f \tag{V.7}$$

and

$$\langle (P \otimes id)\delta_q(M \otimes M) \mid id \otimes f \rangle_2 = \langle \delta_q(M \otimes M) \mid id \otimes P^* f \rangle_2 = \mathcal{F} \tag{V.8}$$

Here the inner product $\langle \cdot \mid \cdot \rangle_2$ which is defined as

$$\langle x_1 \otimes x_2 \mid y_1 \otimes y_2 \rangle_2 = x_1 y_1 \phi(x_2 \tau^{-1}(y_2)); \quad x_1, x_2, y_1, y_2 \in M \tag{V.9}$$

is a map

$$(M \otimes M) \times (M \otimes M) \to M. \tag{V.10}$$

Before closing this section we like to consider the inhomogeneous equation for a given constant E and operator function $f(M) \in M$

$$(\mathcal{C} - E) \circ \mathcal{F} = f. \tag{V.11}$$

As in the classical case the solution is obtained by using the Green function

$$\mathcal{F} = F_0 + \langle \mathcal{G}_q^E(G \otimes G; E) \mid f \otimes id \rangle_2 \tag{V.12}$$

where F_0 is the complete solution of the homogeneous equation.

6 Kernel on the q-symmetric space M

We introduce the unitary operator in terms of the real time interval t and the center of the enveloping algebra $\mathcal{C}$ as

$$U(t) = e^{it\mathcal{C}} \tag{VI.1}$$

satisfying the semigroup property

$$U(t)U(t') = U(t+t'). \tag{VI.2}$$

The one-point "q-Kernel" is then given by

$$\mathcal{K}_q(\xi, t) = U(t)\delta_q(\xi) \tag{VI.3}$$

Inserting the representation of the one-point q-delta function from (IV.3) we obtain

$$\mathcal{K}_q(\xi, t) = \sum_{l=0}^{\infty} [l + \frac{1}{2}] e^{it[l+\frac{1}{2}]^2} d^l_{0,0}(\xi) \tag{VI.4}$$

which is connected to the one-point "Green function" through the relation

$$\mathcal{K}_q(\xi, t) = \int_{-\infty}^{\infty} dE e^{itE} \mathcal{G}^E_q(\xi). \tag{VI.5}$$

It is obvious that the above Kernel satisfies the equation

$$(i\partial_t + \mathcal{C})\mathcal{K}_q(\xi, t) = 0. \tag{VI.6}$$

The two-point q-Kernel is defined in a manner parallel to the definition of the one-point "q-Kernel" as

$$\mathcal{K}_q(M \otimes M, t) = (U(t) \otimes id)\delta_q(M \otimes M) \tag{VI.7}$$

Inserting the representation of the two-point q-delta function of (V.4) into the above equation we have

$$\mathcal{K}_q(M \otimes M, t) = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} e^{it[l+\frac{1}{2}]^2} [l + \frac{1}{2}] \tau(d^l_{0,j}(M)) \otimes \overline{d^l_{0,j}(M)}. \tag{VI.8}$$

The triple invariant product $\langle \cdot \mid \cdot \rangle_3$ defined by

$$\langle x_1 \otimes x_2 \otimes x_3 \mid y_1 \otimes y_2 \otimes y_3 \rangle_3 = x_1 y_1 \otimes x_3 y_3 \phi(x_2 \tau^{-1}(y_2)) \tag{VI.9}$$

which is the map

$$(M \otimes M \otimes M) \times (M \otimes M \otimes M) \to M \otimes M \tag{VI.10}$$

enables us the derivation of the important property of $K_q(M \otimes M, t)$:

$$\langle \mathcal{K}_q(M \otimes M, t) \otimes 1 \mid 1 \otimes \mathcal{K}_q(M \otimes M, t') \rangle_3 = \mathcal{K}_q(M \otimes M, t + t'). \quad \text{(VI.11)}$$

Using this property we can introduce the path integral representation for the Kernel on the non-commutative space. Indeed from (VI.11) follows

$$\mathcal{K}_q(M \otimes M, T) = \langle \mathcal{K}_q(M \otimes M, T/2) \otimes 1 \mid 1 \otimes \mathcal{K}_q(M \otimes M, T/2) \rangle_3 \quad \text{(VI.12)}$$

and

$$\mathcal{K}_q(M \otimes M, T/2) = \langle \mathcal{K}_q(M \otimes M, T/4) \otimes 1 \mid 1 \otimes \mathcal{K}_q(M \otimes M, T/4) \rangle_3 \quad \text{(VI.13)}$$

Inserting (III.13) into (III.12) and using the short hand notation $\mathcal{K}_q(T) = \mathcal{K}_q(M \otimes M, T)$ we get

$$\mathcal{K}_q(T) = \langle \langle \mathcal{K}_q(T/4) \otimes 1 \mid 1 \otimes \mathcal{K}_q(T/4) \rangle_3 \otimes 1 \mid \langle \mathcal{K}_q(T/4) \otimes 1 \mid 1 \otimes \mathcal{K}_q(T/4) \rangle_3 \otimes 1 \rangle_3 \quad \text{(VI.14)}$$

Continuing the above process n times and taking the limit $n \to \infty$ we arrive at a path integral formula:

$$\mathcal{K}_q(M \otimes M, T) = \lim_{n \to \infty} \{ (\mathcal{K}_q(M \otimes M, T/2^n) \}_n \quad \text{(VI.15)}$$

where $\{\cdot\}_n$ stands for n times repeated $\langle \cdot \rangle_3$ map.

7 Green function for the massive free scalar field on the q-Einstein space $R^1 \times M$

For the commutative translation group parametrized by t, the commutative and co-commutative Hopf algebra of $Fun(t)$ is given by

$$\delta t = 1 \otimes t + t \otimes 1, \quad S(t) = -t, \quad \varepsilon(t) = 0. \quad \text{(VII.1)}$$

The one-point Kernel for the free particle motion on (t, s) "space-time" is the usual one

$$\mathcal{K}(t, s) = (-4i\pi s)^{-1/2} e^{-t^2/4s}. \quad \text{(VII.2)}$$

The Kernel on $(R^1 \times M, s)$ is expressed as

$$\mathcal{K}(\xi, t, s) = \mathcal{K}(t, s) \mathcal{K}_q(\xi, s) \quad \text{(VII.3)}$$

where $\mathcal{K}_q(\xi, s)$ is given by (VI.4). Using the Schwinger-DeWitt representation

$$\mathcal{G}(\xi, t; m^2) = -i\theta(t) \int_0^\infty e^{-im^2 s} \mathcal{K}(\xi, t, s) ds \qquad \text{(VII.4)}$$

with $Im(m^2) < 0$ and $\theta(t)$ being the step function we get the one-point "Green function" over $R^1 \times M$ for the scalar field with the mass m. Performing the integration over ds we obtain

$$\mathcal{G}(\xi, t; m^2) = \theta(t) \sum_{n=0}^{\infty} e^{it\sqrt{[n+1/2]_q + m^2}} \frac{[n+1/2]_q}{\sqrt{[n+1/2]_q^2 + m^2}} d_{00}^n(\xi). \qquad \text{(VII.5)}$$

The above Green function satisfies

$$(\partial_t^2 - \mathcal{C} + m^2)\mathcal{G}(\xi, t; m^2) = \delta(t)\delta_q(\xi). \qquad \text{(VII.6)}$$

Following the procedure of Sec. 5 we obtain the invariant Green function on the space $R^1 \times M$ depending on two points

$$\mathcal{G}(Y \otimes Y, ; m^2) = (\tau \otimes S)\Delta\mathcal{G}(\xi, t; m^2), \qquad \text{(VII.7)}$$

where $Y = t \times M$ and the operations Δ and τ on t are given by $\Delta t = \delta t$ and $\tau(t) = t$. The Green function (VII.7) satisfies

$$(id \otimes (\partial_t^2 - \mathcal{C} + m^2))\mathcal{G}(Y \otimes Y; m^2) = \delta(t \otimes 1 - 1 \otimes t)\delta_q(M \otimes M) \qquad \text{(VII.8)}$$

Appendix

A. Hopf Algebra $A = Pol(SU(2)_q)$

The algebra of polynomials $A = A(SU_q(2))$ form the $*$Hopf algebra or real quantum group. The coordinate functions π_{ij} are given by

$$\pi_{ij} d^{1/2} = \pi_{ij} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = d^{1/2}_{ij} \tag{A.1}$$

where $d^{1/2}$ is the matrix of the unitary irreducible corepresentation of the Hopf algebra A. The coproduct Δ, counit ε and antipode S acts as

$$\Delta \circ d^{1/2}_{ij} = d^{1/2}_{ik} \otimes d^{1/2}_{kj} \tag{A.2}$$

$$S(d^{1/2}) = \begin{pmatrix} d & -qb \\ -q^{-1}c & d \end{pmatrix} \tag{A.3}$$

$$\varepsilon \circ d^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.4}$$

The $*$-operation is

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix} \tag{A.5}$$

B. Harmonic Analysis on the Coset Space $M = A/K$

i. The Cartan Decomposition of A.

The quantum group $K = Pol(U(1))$ is the Hopf algebra with coordinate functions $t,\ t^{-1}$

$$\Delta_U \circ t^{\pm} = t^{\pm} \otimes t^{\pm}, \quad S_U \circ t^{\pm} = t^{\mp}, \quad \varepsilon_U \circ t^{\pm} = 1 \tag{B.1}$$

The Hopf algebra K is the subalgebra of the quantum group A defined by the homomorphism

$$\psi_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \tag{B.2}$$

The left and right unitary co-representation of K in A is given by homomorphisms

$$L_u = (\psi_k \otimes id) \circ \Delta, \quad R_u = (id \otimes \psi_k) \circ \Delta. \tag{B.3}$$

The subspaces $A[j, i]$; $j, i \in Z$ defined by

$$A[j, i] = \{x \in A : L_u \circ x = t^j \otimes x; R_u \circ x = x \otimes t^i\} \tag{B.4}$$

form the basis of the Hopf algebra A

$$A = \sum_{j,i \in Z} \oplus A[j, i]. \tag{B.5}$$

The quantum coset space $M = A/K$ and two-sided coset space $H = K/A/K$ are the subspaces of A

$$M = \sum_{j, \in Z} \oplus A[0, j], \qquad H = A[0, 0]. \tag{B.6}$$

ii.Harmonic analysis on M

The irreducible co-representation of A is constructed in the space V of homogeneous polynomials of degree l.

$$T : \Delta V = V \otimes A \tag{B.7}$$

The basis in V is generated by the elements

$$e^l_j = a^{l+j} b^{l-j}. \tag{B.8}$$

The matrix elements of the irreducible co-representation of the $*$Hopf algebra are given in terms of $\xi = -q^{-1}bc \in H$ by

$$D^l_{0j}(M) = \lambda^l_j \phi_{21}(q^{2(j-l)}, q^{2(j+l+1)} q^{2(j+l)} \mid q^2, q\xi) c^j d^j; \quad j = 0, 1, ..., l \tag{B.9}$$

and

$$D^l_{0j}(M) = \lambda^l_j a^{-j} b^{-j} \phi_{21}(q^{2(-j-l)}, q^{2(-j+l+1)} q^{2(-j+l)} \mid q^2, q\xi); \;\; j = -l, -l+1, ..., 0 \tag{B.10}$$

Here ϕ_{21} is the q-hypergeometric function [6], and λ^l_j is defined as

$$\lambda^l_j = q^{|j|(|j|-l)} \begin{bmatrix} l \\ \mid j \mid \end{bmatrix}^{1/2}_{q^2} \begin{bmatrix} l- \mid j \mid \\ \mid j \mid \end{bmatrix}^{1/2}_{q^2} \tag{B.11}$$

The co-representation (B.7) is unitary with respect to the scalar product

$$\langle x \mid y \rangle = \psi(x^* y), \qquad x, y \in V \tag{B.12}$$

where ψ is the invariant integral on A

$$\Psi(z) = \int_0^1 d\xi_q \mathcal{P}(z), \qquad z \in A \tag{B.13}$$

and $\mathcal{P}$ is the projection operator $\mathcal{P}A[i,j] \to A[0,0]$. With respect to the invariant integral (B.13) the matrix elements D^l_{0j} satisfy the orthogonality condition

$$\langle D^l_{0i}(M) \mid D^k_{0j}(M) \rangle = [l + 1/2]^{-1} \delta_{ij} \delta_{lk}. \tag{B.14}$$

The matrix elements given in (B.9) and (B.10) form the orthogonal complete set of functions over M.The Fourier transform of any square integrable function $f \in M$ is given by

$$f = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} [l + 1/2] f^l_j D^l_{j0}(G) \tag{B.15}$$

where the coefficients f^l_j are

$$f^l_j = \langle f \mid D^l_{j0}(G) \rangle. \tag{B.16}$$

C. The Hopf Algebra $U(su_q(2))$

The Hopf Algebra $U(su_q(2))$ is dual to A. Its generated by the elements

$$\mathcal{E}_\pm, \ k_\pm = q^{\pm H/4} \tag{C.1}$$

satisfying the commutation relations

$$[\mathcal{E}_+, \mathcal{E}_-] = \frac{k_+^2 - k_-^2}{q - q^{-1}}, \quad k_+ k_- = k_- k_+, \quad k_+ \mathcal{E}_+ k_- = q \mathcal{E}_+ \tag{C.2}$$

are the linear functionals on A [6]:

$$\mathcal{E}_1(g) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathcal{E}_{-1}(g) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$k_\pm(g) = \begin{pmatrix} q^{\pm 1/2} & 0 \\ 0 & q^{\mp 1/2} \end{pmatrix} \tag{C.3}$$

The extensions of the functionals (C.1) on A are

$$\mathcal{E}_{\pm}(xy) = \mathcal{E}_{\pm}(x)k^{+}(y) + k^{-}(x)\mathcal{E}_{\pm}(y), \qquad k^{\pm}(xy) = k^{\pm}(x)k^{\pm}(y) \tag{C.4}$$

By means of (C.2) and (C.3) we define the differential form of the co-representation (B.7) as

$$D(\psi)d^l_{0,j} = \psi(d^l_{k,j})d^l_{0,k} \tag{C.5}$$

where ψ is one of the elements (C.1). We then have

$$D(\mathcal{E}_{\pm})d^l_{0j}(M) = ([l+1 \pm j][l \mp j])^{1/2} d^l_{0,j\pm 1}(M) \tag{C.6}$$

$$D(k_{\pm})d^l_{0j}(M) = q^{\mp j} d^l_{0j}(M) \tag{C.7}$$

The center of the Hopf algebra $U(su_q(2))$

$$\mathcal{C} = \mathcal{E}_{-}\mathcal{E}_{+} + (\frac{qk_{-} - q^{-1}k_{+}}{q - q^{-1}})^2 \tag{C.8}$$

satisfies the e-value equation

$$(D(\mathcal{C}) - [l+1/2]^2)d^l_{0j} = 0. \tag{C.9}$$

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